Exercise 3

Use Table A.7-2 to write down directly the following quantities in cylindrical coordinates:

(a) $(\nabla \cdot \rho \mathbf{v})$, where ρ is a scalar **(b)** $[\nabla \cdot \rho \mathbf{v}\mathbf{v}]_r$, where ρ is a scalar
(c) $[\nabla \cdot p\boldsymbol{\delta}]_\theta$, where p is a scalar **(d)** $(\nabla \cdot [\boldsymbol{\tau} \cdot \mathbf{v}])$
(e) $[\mathbf{v} \cdot \nabla \mathbf{v}]_\theta$ **(f)** $\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger$

Solution

In cylindrical coordinates the nabla operator is

$$\nabla = \boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z},$$

a general vector in the coordinate system is

$$\mathbf{v} = \boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z,$$

the unit tensor is

$$\boldsymbol{\delta} = \boldsymbol{\delta}_r\boldsymbol{\delta}_r + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta + \boldsymbol{\delta}_z\boldsymbol{\delta}_z,$$

and a general second-order tensor in the coordinate system is

$$\begin{aligned}\boldsymbol{\tau} = {} & \boldsymbol{\delta}_r\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\tau_{r\theta} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\tau_{rz} \\ & + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\tau_{\theta z} \\ & + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\tau_{zr} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\tau_{z\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\tau_{zz}.\end{aligned}$$

The partial derivatives of $\boldsymbol{\delta}_r$, $\boldsymbol{\delta}_\theta$, and $\boldsymbol{\delta}_z$ in cylindrical coordinates are given by equations A.7-1, A.7-2, and A.7-3,

$$\frac{\partial \boldsymbol{\delta}_r}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{\delta}_z}{\partial r} = 0 \tag{A.7-1}$$

$$\frac{\partial \boldsymbol{\delta}_r}{\partial \theta} = \boldsymbol{\delta}_\theta \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta} = -\boldsymbol{\delta}_r \qquad \frac{\partial \boldsymbol{\delta}_z}{\partial \theta} = 0 \tag{A.7-2}$$

$$\frac{\partial \boldsymbol{\delta}_r}{\partial z} = 0 \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial z} = 0 \qquad \frac{\partial \boldsymbol{\delta}_z}{\partial z} = 0. \tag{A.7-3}$$

Part (a)

$$\begin{aligned}\nabla \cdot \rho \mathbf{v} &= \left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right) \cdot \rho(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \\ &= \boldsymbol{\delta}_r \cdot \frac{\partial}{\partial r}(\boldsymbol{\delta}_r \rho v_r + \boldsymbol{\delta}_\theta \rho v_\theta + \boldsymbol{\delta}_z \rho v_z) \\ &\quad + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \frac{\partial}{\partial \theta}(\boldsymbol{\delta}_r \rho v_r + \boldsymbol{\delta}_\theta \rho v_\theta + \boldsymbol{\delta}_z \rho v_z) \\ &\quad + \boldsymbol{\delta}_z \cdot \frac{\partial}{\partial z}(\boldsymbol{\delta}_r \rho v_r + \boldsymbol{\delta}_\theta \rho v_\theta + \boldsymbol{\delta}_z \rho v_z)\end{aligned}$$

Apply the product rule.

$$\begin{aligned}\nabla \cdot \rho \mathbf{v} = \boldsymbol{\delta}_r \cdot \left[\frac{\partial \boldsymbol{\delta}_r}{\partial r}\rho v_r + \boldsymbol{\delta}_r \frac{\partial}{\partial r}(\rho v_r) + \frac{\partial \boldsymbol{\delta}_\theta}{\partial r}\rho v_\theta + \boldsymbol{\delta}_\theta \frac{\partial}{\partial r}(\rho v_\theta) + \frac{\partial \boldsymbol{\delta}_z}{\partial r}\rho v_z + \boldsymbol{\delta}_z \frac{\partial}{\partial r}(\rho v_z)\right] \\ + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \left[\frac{\partial \boldsymbol{\delta}_r}{\partial \theta}\rho v_r + \boldsymbol{\delta}_r \frac{\partial}{\partial \theta}(\rho v_r) + \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta}\rho v_\theta + \boldsymbol{\delta}_\theta \frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial \boldsymbol{\delta}_z}{\partial \theta}\rho v_z + \boldsymbol{\delta}_z \frac{\partial}{\partial \theta}(\rho v_z)\right] \\ + \boldsymbol{\delta}_z \cdot \left[\frac{\partial \boldsymbol{\delta}_r}{\partial z}\rho v_r + \boldsymbol{\delta}_r \frac{\partial}{\partial z}(\rho v_r) + \frac{\partial \boldsymbol{\delta}_\theta}{\partial z}\rho v_\theta + \boldsymbol{\delta}_\theta \frac{\partial}{\partial z}(\rho v_\theta) + \frac{\partial \boldsymbol{\delta}_z}{\partial z}\rho v_z + \boldsymbol{\delta}_z \frac{\partial}{\partial z}(\rho v_z)\right]\end{aligned}$$

Use equations A.7-1, A.7-2, and A.7-3 here.

$$\begin{aligned}= \boldsymbol{\delta}_r \cdot \left[\boldsymbol{\delta}_r \frac{\partial}{\partial r}(\rho v_r) + \boldsymbol{\delta}_\theta \frac{\partial}{\partial r}(\rho v_\theta) + \boldsymbol{\delta}_z \frac{\partial}{\partial r}(\rho v_z)\right] \\ + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \left[\boldsymbol{\delta}_\theta \rho v_r + \boldsymbol{\delta}_r \frac{\partial}{\partial \theta}(\rho v_r) - \boldsymbol{\delta}_r \rho v_\theta + \boldsymbol{\delta}_\theta \frac{\partial}{\partial \theta}(\rho v_\theta) + \boldsymbol{\delta}_z \frac{\partial}{\partial \theta}(\rho v_z)\right] \\ + \boldsymbol{\delta}_z \cdot \left[\boldsymbol{\delta}_r \frac{\partial}{\partial z}(\rho v_r) + \boldsymbol{\delta}_\theta \frac{\partial}{\partial z}(\rho v_\theta) + \boldsymbol{\delta}_z \frac{\partial}{\partial z}(\rho v_z)\right]\end{aligned}$$

Evaluate the dot products.

$$\begin{aligned}= \frac{\partial}{\partial r}(\rho v_r) \\ + \frac{1}{r}\left[\rho v_r + \frac{\partial}{\partial \theta}(\rho v_\theta)\right] \\ + \frac{\partial}{\partial z}(\rho v_z)\end{aligned}$$

Distribute $1/r$.

$$= \frac{\partial}{\partial r}(\rho v_r) + \frac{1}{r}\rho v_r + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z)$$

Factor $1/r$ from the first two terms.

$$= \frac{1}{r}\left[r\frac{\partial}{\partial r}(\rho v_r) + \rho v_r\right] + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z)$$

Use the product rule to write the expression compactly and obtain the final result for the divergence of $\rho\mathbf{v}$ in cylindrical coordinates.

$$\nabla \cdot \rho \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z)$$

Part (b)

$$\begin{aligned}\nabla \cdot \rho \mathbf{v}\mathbf{v} &= \left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right) \cdot \rho(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z)(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \\ &= \left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right) \cdot \rho(\boldsymbol{\delta}_r\boldsymbol{\delta}_r v_r^2 + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta v_r v_\theta + \boldsymbol{\delta}_r\boldsymbol{\delta}_z v_r v_z \\ &\qquad + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r v_\theta v_r + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta v_\theta^2 + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z v_\theta v_z \\ &\qquad + \boldsymbol{\delta}_z\boldsymbol{\delta}_r v_z v_r + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta v_z v_\theta + \boldsymbol{\delta}_z\boldsymbol{\delta}_z v_z^2)\end{aligned}$$

$$\begin{aligned}\nabla\cdot\rho\mathbf{v}\mathbf{v} = \boldsymbol{\delta}_r\cdot\frac{\partial}{\partial r}(&\boldsymbol{\delta}_r\boldsymbol{\delta}_r\rho v_r^2+\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\rho v_r v_\theta+\boldsymbol{\delta}_r\boldsymbol{\delta}_z\rho v_r v_z+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\rho v_\theta v_r+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\rho v_\theta^2+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\rho v_\theta v_z\\ &+\boldsymbol{\delta}_z\boldsymbol{\delta}_r\rho v_z v_r+\boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\rho v_z v_\theta+\boldsymbol{\delta}_z\boldsymbol{\delta}_z\rho v_z^2)\\ +\frac{\boldsymbol{\delta}_\theta}{r}\cdot\frac{\partial}{\partial\theta}(&\boldsymbol{\delta}_r\boldsymbol{\delta}_r\rho v_r^2+\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\rho v_r v_\theta+\boldsymbol{\delta}_r\boldsymbol{\delta}_z\rho v_r v_z+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\rho v_\theta v_r+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\rho v_\theta^2+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\rho v_\theta v_z\\ &+\boldsymbol{\delta}_z\boldsymbol{\delta}_r\rho v_z v_r+\boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\rho v_z v_\theta+\boldsymbol{\delta}_z\boldsymbol{\delta}_z\rho v_z^2)\\ +\boldsymbol{\delta}_z\cdot\frac{\partial}{\partial z}(&\boldsymbol{\delta}_r\boldsymbol{\delta}_r\rho v_r^2+\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\rho v_r v_\theta+\boldsymbol{\delta}_r\boldsymbol{\delta}_z\rho v_r v_z+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\rho v_\theta v_r+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\rho v_\theta^2+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\rho v_\theta v_z\\ &+\boldsymbol{\delta}_z\boldsymbol{\delta}_r\rho v_z v_r+\boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\rho v_z v_\theta+\boldsymbol{\delta}_z\boldsymbol{\delta}_z\rho v_z^2)\end{aligned}$$

Apply the product rule.

$$\begin{aligned}&=\boldsymbol{\delta}_r\cdot\left[\frac{\partial\boldsymbol{\delta}_r}{\partial r}\boldsymbol{\delta}_r\rho v_r^2+\boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_r}{\partial r}\rho v_r^2+\boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial}{\partial r}(\rho v_r^2)+\frac{\partial\boldsymbol{\delta}_r}{\partial r}\boldsymbol{\delta}_\theta\rho v_r v_\theta+\boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_\theta}{\partial r}\rho v_r v_\theta+\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial}{\partial r}(\rho v_r v_\theta)+\cdots\right]\\ &\quad+\frac{\boldsymbol{\delta}_\theta}{r}\cdot\left[\frac{\partial\boldsymbol{\delta}_r}{\partial\theta}\boldsymbol{\delta}_r\rho v_r^2+\boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_r}{\partial\theta}\rho v_r^2+\boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial}{\partial\theta}(\rho v_r^2)+\frac{\partial\boldsymbol{\delta}_r}{\partial\theta}\boldsymbol{\delta}_\theta\rho v_r v_\theta+\boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta}\rho v_r v_\theta+\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}(\rho v_r v_\theta)+\cdots\right]\\ &\quad+\boldsymbol{\delta}_z\cdot\left[\frac{\partial\boldsymbol{\delta}_r}{\partial z}\boldsymbol{\delta}_r\rho v_r^2+\boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_r}{\partial z}\rho v_r^2+\boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial}{\partial z}(\rho v_r^2)+\frac{\partial\boldsymbol{\delta}_r}{\partial z}\boldsymbol{\delta}_\theta\rho v_r v_\theta+\boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_\theta}{\partial z}\rho v_r v_\theta+\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial}{\partial z}(\rho v_r v_\theta)+\cdots\right]\end{aligned}$$

Use equations A.7-1, A.7-2, and A.7-3 and evaluate the dot products.

$$\begin{aligned}&=\boldsymbol{\delta}_r\frac{\partial}{\partial r}(\rho v_r^2)+\boldsymbol{\delta}_\theta\frac{\partial}{\partial r}(\rho v_r v_\theta)+\boldsymbol{\delta}_z\frac{\partial}{\partial r}(\rho v_r v_z)\\ &\quad+\frac{1}{r}\left[\boldsymbol{\delta}_r\rho v_r^2+\boldsymbol{\delta}_\theta\rho v_r v_\theta+\boldsymbol{\delta}_z\rho v_r v_z+\boldsymbol{\delta}_\theta\rho v_\theta v_r+\boldsymbol{\delta}_r\frac{\partial}{\partial\theta}(\rho v_\theta v_r)-\boldsymbol{\delta}_r\rho v_\theta^2+\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}(\rho v_\theta^2)+\boldsymbol{\delta}_z\frac{\partial}{\partial\theta}(\rho v_\theta v_z)\right]\\ &\quad+\boldsymbol{\delta}_r\frac{\partial}{\partial z}(\rho v_z v_r)+\boldsymbol{\delta}_\theta\frac{\partial}{\partial z}(\rho v_z v_\theta)+\boldsymbol{\delta}_z\frac{\partial}{\partial z}(\rho v_z^2)\end{aligned}$$

We only care about the r-component of $\nabla\cdot\rho\mathbf{v}\mathbf{v}$, so only the terms with $\boldsymbol{\delta}_r$ remain.

$$\begin{aligned}[\nabla\cdot\rho\mathbf{v}\mathbf{v}]_r&=\frac{\partial}{\partial r}(\rho v_r^2)+\frac{1}{r}\left[\rho v_r^2+\frac{\partial}{\partial\theta}(\rho v_\theta v_r)-\rho v_\theta^2\right]+\frac{\partial}{\partial z}(\rho v_z v_r)\\ &=\frac{\partial}{\partial r}(\rho v_r^2)+\frac{\rho v_r^2}{r}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta v_r)-\frac{\rho v_\theta^2}{r}+\frac{\partial}{\partial z}(\rho v_z v_r)\\ &=\frac{1}{r}\left[r\frac{\partial}{\partial r}(\rho v_r^2)+\rho v_r^2\right]+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta v_r)-\frac{\rho v_\theta^2}{r}+\frac{\partial}{\partial z}(\rho v_z v_r)\end{aligned}$$

Therefore,

$$[\nabla\cdot\rho\mathbf{v}\mathbf{v}]_r=\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r^2)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta v_r)-\frac{\rho v_\theta^2}{r}+\frac{\partial}{\partial z}(\rho v_z v_r).$$

Part (c)

$$
\begin{aligned}
\nabla \cdot p\boldsymbol{\delta} &= \left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right) \cdot p(\boldsymbol{\delta}_r\boldsymbol{\delta}_r + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta + \boldsymbol{\delta}_z\boldsymbol{\delta}_z) \\
&= \left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right) \cdot (\boldsymbol{\delta}_r\boldsymbol{\delta}_r p + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta p + \boldsymbol{\delta}_z\boldsymbol{\delta}_z p) \\
&= \boldsymbol{\delta}_r \cdot \frac{\partial}{\partial r}(\boldsymbol{\delta}_r\boldsymbol{\delta}_r p + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta p + \boldsymbol{\delta}_z\boldsymbol{\delta}_z p) \\
&\quad + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \frac{\partial}{\partial \theta}(\boldsymbol{\delta}_r\boldsymbol{\delta}_r p + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta p + \boldsymbol{\delta}_z\boldsymbol{\delta}_z p) \\
&\quad + \boldsymbol{\delta}_z \cdot \frac{\partial}{\partial z}(\boldsymbol{\delta}_r\boldsymbol{\delta}_r p + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta p + \boldsymbol{\delta}_z\boldsymbol{\delta}_z p)
\end{aligned}
$$

Apply the product rule.

$$
\begin{aligned}
&= \boldsymbol{\delta}_r \cdot \left(\frac{\partial \boldsymbol{\delta}_r}{\partial r}\boldsymbol{\delta}_r p + \boldsymbol{\delta}_r \frac{\partial \boldsymbol{\delta}_r}{\partial r} p + \boldsymbol{\delta}_r\boldsymbol{\delta}_r \frac{\partial p}{\partial r} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial r}\boldsymbol{\delta}_\theta p + \boldsymbol{\delta}_\theta \frac{\partial \boldsymbol{\delta}_\theta}{\partial r} p + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta \frac{\partial p}{\partial r} + \frac{\partial \boldsymbol{\delta}_z}{\partial r}\boldsymbol{\delta}_z p + \boldsymbol{\delta}_z \frac{\partial \boldsymbol{\delta}_z}{\partial r} p + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \frac{\partial p}{\partial r}\right) \\
&\quad + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \left(\frac{\partial \boldsymbol{\delta}_r}{\partial \theta}\boldsymbol{\delta}_r p + \boldsymbol{\delta}_r \frac{\partial \boldsymbol{\delta}_r}{\partial \theta} p + \boldsymbol{\delta}_r\boldsymbol{\delta}_r \frac{\partial p}{\partial \theta} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta}\boldsymbol{\delta}_\theta p + \boldsymbol{\delta}_\theta \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta} p + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta \frac{\partial p}{\partial \theta} + \frac{\partial \boldsymbol{\delta}_z}{\partial \theta}\boldsymbol{\delta}_z p + \boldsymbol{\delta}_z \frac{\partial \boldsymbol{\delta}_z}{\partial \theta} p + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \frac{\partial p}{\partial \theta}\right) \\
&\quad + \boldsymbol{\delta}_z \cdot \left(\frac{\partial \boldsymbol{\delta}_r}{\partial z}\boldsymbol{\delta}_r p + \boldsymbol{\delta}_r \frac{\partial \boldsymbol{\delta}_r}{\partial z} p + \boldsymbol{\delta}_r\boldsymbol{\delta}_r \frac{\partial p}{\partial z} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial z}\boldsymbol{\delta}_\theta p + \boldsymbol{\delta}_\theta \frac{\partial \boldsymbol{\delta}_\theta}{\partial z} p + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta \frac{\partial p}{\partial z} + \frac{\partial \boldsymbol{\delta}_z}{\partial z}\boldsymbol{\delta}_z p + \boldsymbol{\delta}_z \frac{\partial \boldsymbol{\delta}_z}{\partial z} p + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \frac{\partial p}{\partial z}\right)
\end{aligned}
$$

Use equations A.7-1, A.7-2, and A.7-3 here.

$$
\begin{aligned}
&= \boldsymbol{\delta}_r \cdot \left(\boldsymbol{\delta}_r\boldsymbol{\delta}_r \frac{\partial p}{\partial r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta \frac{\partial p}{\partial r} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \frac{\partial p}{\partial r}\right) \\
&\quad + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \left(\cancel{\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r p} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta p + \boldsymbol{\delta}_r\boldsymbol{\delta}_r \frac{\partial p}{\partial \theta} - \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta p - \cancel{\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r p} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta \frac{\partial p}{\partial \theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \frac{\partial p}{\partial \theta}\right) \\
&\quad + \boldsymbol{\delta}_z \cdot \left(\boldsymbol{\delta}_r\boldsymbol{\delta}_r \frac{\partial p}{\partial z} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta \frac{\partial p}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \frac{\partial p}{\partial z}\right)
\end{aligned}
$$

Evaluate the dot products. The unit vector in front of the parentheses is dotted with the first unit vector in each dyad.

$$
\begin{aligned}
&= \boldsymbol{\delta}_r \frac{\partial p}{\partial r} \\
&\quad + \frac{1}{r}\left(\boldsymbol{\delta}_\theta \frac{\partial p}{\partial \theta}\right) \\
&\quad + \boldsymbol{\delta}_z \frac{\partial p}{\partial z}
\end{aligned}
$$

Hence,

$$\nabla \cdot p\boldsymbol{\delta} = \boldsymbol{\delta}_r \frac{\partial p}{\partial r} + \frac{\boldsymbol{\delta}_\theta}{r}\frac{\partial p}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial p}{\partial z}.$$

The θ-component is what we care about. Therefore,

$$[\nabla \cdot p\boldsymbol{\delta}]_\theta = \frac{1}{r}\frac{\partial p}{\partial \theta}.$$

Part (d)

The strategy here is to find $\boldsymbol{\tau} \cdot \mathbf{v}$ first and then take the divergence of it.

$$\begin{aligned}
\boldsymbol{\tau} \cdot \mathbf{v} &= (\boldsymbol{\delta}_r\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\tau_{r\theta} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\tau_{rz} \\
&\qquad + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\tau_{\theta z} \\
&\qquad + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\tau_{zr} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\tau_{z\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\tau_{zz}) \cdot (\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \\
&= [(\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_\theta\tau_{\theta r} + \boldsymbol{\delta}_z\tau_{zr})\boldsymbol{\delta}_r \\
&\qquad + (\boldsymbol{\delta}_r\tau_{r\theta} + \boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_z\tau_{z\theta})\boldsymbol{\delta}_\theta \\
&\qquad + (\boldsymbol{\delta}_r\tau_{rz} + \boldsymbol{\delta}_\theta\tau_{\theta z} + \boldsymbol{\delta}_z\tau_{zz})\boldsymbol{\delta}_z] \cdot (\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \\
&= (\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_\theta\tau_{\theta r} + \boldsymbol{\delta}_z\tau_{zr})v_r \\
&\qquad + (\boldsymbol{\delta}_r\tau_{r\theta} + \boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_z\tau_{z\theta})v_\theta \\
&\qquad + (\boldsymbol{\delta}_r\tau_{rz} + \boldsymbol{\delta}_\theta\tau_{\theta z} + \boldsymbol{\delta}_z\tau_{zz})v_z \\
&= \boldsymbol{\delta}_r(\tau_{rr}v_r + \tau_{r\theta}v_\theta + \tau_{rz}v_z) + \boldsymbol{\delta}_\theta(\tau_{\theta r}v_r + \tau_{\theta\theta}v_\theta + \tau_{\theta z}v_z) + \boldsymbol{\delta}_z(\tau_{zr}v_r + \tau_{z\theta}v_\theta + \tau_{zz}v_z)
\end{aligned}$$

Now we're ready to take the divergence.

$$\begin{aligned}
\nabla \cdot [\boldsymbol{\tau} \cdot \mathbf{v}] &= \left(\boldsymbol{\delta}_r\frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z\frac{\partial}{\partial z}\right) \cdot [\boldsymbol{\delta}_r(\tau_{rr}v_r + \tau_{r\theta}v_\theta + \tau_{rz}v_z) \\
&\qquad + \boldsymbol{\delta}_\theta(\tau_{\theta r}v_r + \tau_{\theta\theta}v_\theta + \tau_{\theta z}v_z) \\
&\qquad + \boldsymbol{\delta}_z(\tau_{zr}v_r + \tau_{z\theta}v_\theta + \tau_{zz}v_z)] \\
&= \boldsymbol{\delta}_r \cdot \frac{\partial}{\partial r}[\boldsymbol{\delta}_r(\tau_{rr}v_r + \tau_{r\theta}v_\theta + \tau_{rz}v_z) + \boldsymbol{\delta}_\theta(\tau_{\theta r}v_r + \tau_{\theta\theta}v_\theta + \tau_{\theta z}v_z) + \boldsymbol{\delta}_z(\tau_{zr}v_r + \tau_{z\theta}v_\theta + \tau_{zz}v_z)] \\
&\quad + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \frac{\partial}{\partial \theta}[\boldsymbol{\delta}_r(\tau_{rr}v_r + \tau_{r\theta}v_\theta + \tau_{rz}v_z) + \boldsymbol{\delta}_\theta(\tau_{\theta r}v_r + \tau_{\theta\theta}v_\theta + \tau_{\theta z}v_z) + \boldsymbol{\delta}_z(\tau_{zr}v_r + \tau_{z\theta}v_\theta + \tau_{zz}v_z)] \\
&\quad + \boldsymbol{\delta}_z \cdot \frac{\partial}{\partial z}[\boldsymbol{\delta}_r(\tau_{rr}v_r + \tau_{r\theta}v_\theta + \tau_{rz}v_z) + \boldsymbol{\delta}_\theta(\tau_{\theta r}v_r + \tau_{\theta\theta}v_\theta + \tau_{\theta z}v_z) + \boldsymbol{\delta}_z(\tau_{zr}v_r + \tau_{z\theta}v_\theta + \tau_{zz}v_z)]
\end{aligned}$$

Apply the product rule.

$$\begin{aligned}
&= \boldsymbol{\delta}_r \cdot \left[\frac{\partial \boldsymbol{\delta}_r}{\partial r}(\tau_{rr}v_r + \tau_{r\theta}v_\theta + \tau_{rz}v_z) + \boldsymbol{\delta}_r\frac{\partial}{\partial r}(\tau_{rr}v_r + \tau_{r\theta}v_\theta + \tau_{rz}v_z) + \frac{\partial \boldsymbol{\delta}_\theta}{\partial r}(\tau_{\theta r}v_r + \tau_{\theta\theta}v_\theta + \tau_{\theta z}v_z)\right. \\
&\qquad \left. + \boldsymbol{\delta}_\theta\frac{\partial}{\partial r}(\tau_{\theta r}v_r + \tau_{\theta\theta}v_\theta + \tau_{\theta z}v_z) + \frac{\partial \boldsymbol{\delta}_z}{\partial r}(\tau_{zr}v_r + \tau_{z\theta}v_\theta + \tau_{zz}v_z) + \boldsymbol{\delta}_z\frac{\partial}{\partial r}(\tau_{zr}v_r + \tau_{z\theta}v_\theta + \tau_{zz}v_z)\right] \\
&\quad + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \left[\frac{\partial \boldsymbol{\delta}_r}{\partial \theta}(\tau_{rr}v_r + \tau_{r\theta}v_\theta + \tau_{rz}v_z) + \boldsymbol{\delta}_r\frac{\partial}{\partial \theta}(\tau_{rr}v_r + \tau_{r\theta}v_\theta + \tau_{rz}v_z) + \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta}(\tau_{\theta r}v_r + \tau_{\theta\theta}v_\theta + \tau_{\theta z}v_z)\right. \\
&\qquad \left. + \boldsymbol{\delta}_\theta\frac{\partial}{\partial \theta}(\tau_{\theta r}v_r + \tau_{\theta\theta}v_\theta + \tau_{\theta z}v_z) + \frac{\partial \boldsymbol{\delta}_z}{\partial \theta}(\tau_{zr}v_r + \tau_{z\theta}v_\theta + \tau_{zz}v_z) + \boldsymbol{\delta}_z\frac{\partial}{\partial \theta}(\tau_{zr}v_r + \tau_{z\theta}v_\theta + \tau_{zz}v_z)\right] \\
&\quad + \boldsymbol{\delta}_z \cdot \left[\frac{\partial \boldsymbol{\delta}_r}{\partial z}(\tau_{rr}v_r + \tau_{r\theta}v_\theta + \tau_{rz}v_z) + \boldsymbol{\delta}_r\frac{\partial}{\partial z}(\tau_{rr}v_r + \tau_{r\theta}v_\theta + \tau_{rz}v_z) + \frac{\partial \boldsymbol{\delta}_\theta}{\partial z}(\tau_{\theta r}v_r + \tau_{\theta\theta}v_\theta + \tau_{\theta z}v_z)\right. \\
&\qquad \left. + \boldsymbol{\delta}_\theta\frac{\partial}{\partial z}(\tau_{\theta r}v_r + \tau_{\theta\theta}v_\theta + \tau_{\theta z}v_z) + \frac{\partial \boldsymbol{\delta}_z}{\partial z}(\tau_{zr}v_r + \tau_{z\theta}v_\theta + \tau_{zz}v_z) + \boldsymbol{\delta}_z\frac{\partial}{\partial z}(\tau_{zr}v_r + \tau_{z\theta}v_\theta + \tau_{zz}v_z)\right]
\end{aligned}$$

Use equations A.7-1, A.7-2, and A.7-3 here.

$$\begin{aligned}\nabla\cdot[\boldsymbol{\tau}\cdot\mathbf{v}] = \boldsymbol{\delta}_r\cdot&\left[\boldsymbol{\delta}_r\frac{\partial}{\partial r}(\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z)+\boldsymbol{\delta}_\theta\frac{\partial}{\partial r}(\tau_{\theta r}v_r+\tau_{\theta\theta}v_\theta+\tau_{\theta z}v_z)+\boldsymbol{\delta}_z\frac{\partial}{\partial r}(\tau_{zr}v_r+\tau_{z\theta}v_\theta+\tau_{zz}v_z)\right]\\ &+\frac{\boldsymbol{\delta}_\theta}{r}\cdot\left[\boldsymbol{\delta}_\theta(\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z)+\boldsymbol{\delta}_r\frac{\partial}{\partial\theta}(\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z)-\boldsymbol{\delta}_r(\tau_{\theta r}v_r+\tau_{\theta\theta}v_\theta+\tau_{\theta z}v_z)\right.\\ &\qquad\left.+\boldsymbol{\delta}_\theta\frac{\partial}{\partial\theta}(\tau_{\theta r}v_r+\tau_{\theta\theta}v_\theta+\tau_{\theta z}v_z)+\boldsymbol{\delta}_z\frac{\partial}{\partial\theta}(\tau_{zr}v_r+\tau_{z\theta}v_\theta+\tau_{zz}v_z)\right]\\ &+\boldsymbol{\delta}_z\cdot\left[\boldsymbol{\delta}_r\frac{\partial}{\partial z}(\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z)+\boldsymbol{\delta}_\theta\frac{\partial}{\partial z}(\tau_{\theta r}v_r+\tau_{\theta\theta}v_\theta+\tau_{\theta z}v_z)+\boldsymbol{\delta}_z\frac{\partial}{\partial z}(\tau_{zr}v_r+\tau_{z\theta}v_\theta+\tau_{zz}v_z)\right]\end{aligned}$$

Evaluate the dot products.

$$\begin{aligned}&=\frac{\partial}{\partial r}(\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z)\\ &\quad+\frac{1}{r}\left[(\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z)+\frac{\partial}{\partial\theta}(\tau_{\theta r}v_r+\tau_{\theta\theta}v_\theta+\tau_{\theta z}v_z)\right]\\ &\quad+\frac{\partial}{\partial z}(\tau_{zr}v_r+\tau_{z\theta}v_\theta+\tau_{zz}v_z)\end{aligned}$$

Distribute $1/r$.

$$\begin{aligned}&=\frac{\partial}{\partial r}(\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z)+\frac{1}{r}(\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z)+\frac{1}{r}\frac{\partial}{\partial\theta}(\tau_{\theta r}v_r+\tau_{\theta\theta}v_\theta+\tau_{\theta z}v_z)\\ &\quad+\frac{\partial}{\partial z}(\tau_{zr}v_r+\tau_{z\theta}v_\theta+\tau_{zz}v_z)\end{aligned}$$

Factor $1/r$ from the first two terms.

$$\begin{aligned}&=\frac{1}{r}\left[r\frac{\partial}{\partial r}(\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z)+\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z\right]+\frac{1}{r}\frac{\partial}{\partial\theta}(\tau_{\theta r}v_r+\tau_{\theta\theta}v_\theta+\tau_{\theta z}v_z)\\ &\quad+\frac{\partial}{\partial z}(\tau_{zr}v_r+\tau_{z\theta}v_\theta+\tau_{zz}v_z)\end{aligned}$$

Use the product rule to write the expression compactly. Therefore,

$$\nabla\cdot[\boldsymbol{\tau}\cdot\mathbf{v}]=\frac{1}{r}\frac{\partial}{\partial r}[r(\tau_{rr}v_r+\tau_{r\theta}v_\theta+\tau_{rz}v_z)]+\frac{1}{r}\frac{\partial}{\partial\theta}(\tau_{\theta r}v_r+\tau_{\theta\theta}v_\theta+\tau_{\theta z}v_z)+\frac{\partial}{\partial z}(\tau_{zr}v_r+\tau_{z\theta}v_\theta+\tau_{zz}v_z).$$

Part (e)

$$\begin{aligned}\mathbf{v}\cdot\nabla\mathbf{v}&=(\boldsymbol{\delta}_r v_r+\boldsymbol{\delta}_\theta v_\theta+\boldsymbol{\delta}_z v_z)\cdot\left(\boldsymbol{\delta}_r\frac{\partial}{\partial r}+\boldsymbol{\delta}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\boldsymbol{\delta}_z\frac{\partial}{\partial z}\right)(\boldsymbol{\delta}_r v_r+\boldsymbol{\delta}_\theta v_\theta+\boldsymbol{\delta}_z v_z)\\ &=(\boldsymbol{\delta}_r v_r+\boldsymbol{\delta}_\theta v_\theta+\boldsymbol{\delta}_z v_z)\cdot\left[\boldsymbol{\delta}_r\frac{\partial}{\partial r}(\boldsymbol{\delta}_r v_r+\boldsymbol{\delta}_\theta v_\theta+\boldsymbol{\delta}_z v_z)\right.\\ &\qquad+\frac{\boldsymbol{\delta}_\theta}{r}\frac{\partial}{\partial\theta}(\boldsymbol{\delta}_r v_r+\boldsymbol{\delta}_\theta v_\theta+\boldsymbol{\delta}_z v_z)\\ &\qquad\left.+\boldsymbol{\delta}_z\frac{\partial}{\partial z}(\boldsymbol{\delta}_r v_r+\boldsymbol{\delta}_\theta v_\theta+\boldsymbol{\delta}_z v_z)\right]\end{aligned}$$

Apply the product rule.

$$
\begin{aligned}
= (\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \cdot \Bigg[\boldsymbol{\delta}_r & \left(\frac{\partial \boldsymbol{\delta}_r}{\partial r} v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial r} v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial r} + \frac{\partial \boldsymbol{\delta}_z}{\partial r} v_z + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial r} \right) \\
& + \frac{\boldsymbol{\delta}_\theta}{r} \left(\frac{\partial \boldsymbol{\delta}_r}{\partial \theta} v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial \theta} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta} v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial \theta} + \frac{\partial \boldsymbol{\delta}_z}{\partial \theta} v_z + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial \theta} \right) \\
& + \boldsymbol{\delta}_z \left(\frac{\partial \boldsymbol{\delta}_r}{\partial z} v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial z} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial z} v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial z} + \frac{\partial \boldsymbol{\delta}_z}{\partial z} v_z + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial z} \right) \Bigg]
\end{aligned}
$$

Use equations A.7-1, A.7-2, and A.7-3.

$$
\begin{aligned}
= (\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \cdot \Bigg[\boldsymbol{\delta}_r & \left(\boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial r} \right) \\
& + \frac{\boldsymbol{\delta}_\theta}{r} \left(\boldsymbol{\delta}_\theta v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial \theta} - \boldsymbol{\delta}_r v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial \theta} \right) \\
& + \boldsymbol{\delta}_z \left(\boldsymbol{\delta}_r \frac{\partial v_r}{\partial z} + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial z} + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial z} \right) \Bigg]
\end{aligned}
$$

Evaluate the dot product.

$$
\begin{aligned}
= v_r & \left(\boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial r} \right) \\
& + \frac{v_\theta}{r} \left(\boldsymbol{\delta}_\theta v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial \theta} - \boldsymbol{\delta}_r v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial \theta} \right) \\
& + v_z \left(\boldsymbol{\delta}_r \frac{\partial v_r}{\partial z} + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial z} + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial z} \right)
\end{aligned}
$$

Factor the unit vectors.

$$
\begin{aligned}
= \boldsymbol{\delta}_r \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) + \boldsymbol{\delta}_\theta & \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} \right) \\
& + \boldsymbol{\delta}_z \left(v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right)
\end{aligned}
$$

We only care about the θ-component.

$$
[\mathbf{v} \cdot \nabla \mathbf{v}]_\theta = v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z}
$$

Therefore,

$$
[\mathbf{v} \cdot \nabla \mathbf{v}]_\theta = v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \left(v_r + \frac{\partial v_\theta}{\partial \theta} \right) + v_z \frac{\partial v_\theta}{\partial z}.
$$

Part (f)

We will find an expression for $\nabla \mathbf{v}$ and then use that to determine $(\nabla \mathbf{v})^\dagger$. Once we know both we can determine the sum of the two.

$$
\begin{aligned}
\nabla \mathbf{v} &= \left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right)(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \\
&= \boldsymbol{\delta}_r \frac{\partial}{\partial r}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \\
&\quad + \frac{\boldsymbol{\delta}_\theta}{r}\frac{\partial}{\partial \theta}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \\
&\quad + \boldsymbol{\delta}_z \frac{\partial}{\partial z}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z)
\end{aligned}
$$

Apply the product rule.

$$
\begin{aligned}
&= \boldsymbol{\delta}_r \left(\frac{\partial \boldsymbol{\delta}_r}{\partial r} v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial r} v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial r} + \frac{\partial \boldsymbol{\delta}_z}{\partial r} v_z + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial r}\right) \\
&\quad + \frac{\boldsymbol{\delta}_\theta}{r}\left(\frac{\partial \boldsymbol{\delta}_r}{\partial \theta} v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial \theta} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta} v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial \theta} + \frac{\partial \boldsymbol{\delta}_z}{\partial \theta} v_z + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial \theta}\right) \\
&\quad + \boldsymbol{\delta}_z \left(\frac{\partial \boldsymbol{\delta}_r}{\partial z} v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial z} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial z} v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial z} + \frac{\partial \boldsymbol{\delta}_z}{\partial z} v_z + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial z}\right)
\end{aligned}
$$

Use equations A.7-1, A.7-2, and A.7-3.

$$
\begin{aligned}
&= \boldsymbol{\delta}_r \left(\boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial r}\right) \\
&\quad + \frac{\boldsymbol{\delta}_\theta}{r}\left(\boldsymbol{\delta}_\theta v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial \theta} - \boldsymbol{\delta}_r v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial \theta}\right) \\
&\quad + \boldsymbol{\delta}_z \left(\boldsymbol{\delta}_r \frac{\partial v_r}{\partial z} + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial z} + \boldsymbol{\delta}_z \frac{\partial v_z}{\partial z}\right)
\end{aligned}
$$

Distribute the unit vectors.

$$
\begin{aligned}
&= \boldsymbol{\delta}_r\boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z \frac{\partial v_z}{\partial r} \\
&\quad + \frac{\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) + \frac{\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\boldsymbol{\delta}_\theta\boldsymbol{\delta}_z}{r}\frac{\partial v_z}{\partial \theta} \\
&\quad + \boldsymbol{\delta}_z\boldsymbol{\delta}_r \frac{\partial v_r}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \frac{\partial v_z}{\partial z}
\end{aligned}
$$

Now $(\nabla \mathbf{v})^\dagger$ can be obtained by taking the transpose of this tensor.

$$
\begin{aligned}
(\nabla \mathbf{v})^\dagger &= \boldsymbol{\delta}_r\boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \frac{\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) + \boldsymbol{\delta}_r\boldsymbol{\delta}_z \frac{\partial v_r}{\partial z} \\
&\quad + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r \frac{\partial v_\theta}{\partial r} + \frac{\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z \frac{\partial v_\theta}{\partial z} \\
&\quad + \boldsymbol{\delta}_z\boldsymbol{\delta}_r \frac{\partial v_z}{\partial r} + \frac{\boldsymbol{\delta}_z\boldsymbol{\delta}_\theta}{r}\frac{\partial v_z}{\partial \theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \frac{\partial v_z}{\partial z}
\end{aligned}
$$

The sum is thus

$$
\begin{aligned}
\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger &= \boldsymbol{\delta}_r\boldsymbol{\delta}_r \left(\frac{\partial v_r}{\partial r} + \frac{\partial v_r}{\partial r}\right) + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta \left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right] + \boldsymbol{\delta}_r\boldsymbol{\delta}_z \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \\
&\quad + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r \left[\frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) + \frac{\partial v_\theta}{\partial r}\right] + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta \left[\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\right] + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right) \\
&\quad + \boldsymbol{\delta}_z\boldsymbol{\delta}_r \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \left(\frac{\partial v_z}{\partial z} + \frac{\partial v_z}{\partial z}\right).
\end{aligned}
$$

Therefore,

$$\begin{aligned}\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger = {} & \boldsymbol{\delta}_r\boldsymbol{\delta}_r\left(2\frac{\partial v_r}{\partial r}\right) + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right] + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \\ & + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right] + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\frac{2}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right) \\ & + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right) + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\left(2\frac{\partial v_z}{\partial z}\right).\end{aligned}$$